The condition that (9) contain only derivatives in the characteristic direction is expressed by equality of the ratios

$$\left.\begin{aligned}(\nu_1 u + 2\nu_2 c/(\gamma - 1))/\nu_1 = (\nu_1(\gamma-1)c/2 + \nu_2 u + \nu_3 B_2)/\nu_2 = \\ (\nu_5 u - \nu_3 B_1)/\nu_5 = (\nu_2 b_2^2/B_2 - \nu_5 b_1^2/B_1 + \nu_3 u)/\nu_3 = \\ (\nu_4 u - \nu_2 c^2/\gamma(\gamma-1)c_v)/\nu_4 = dx/dt = u + a\end{aligned}\right\} \quad (10)$$

The solutions for ν_j/ν_1 ($j = 2, 3, 4, 5$) are

$$\left.\begin{aligned}\frac{\nu_2}{\nu_1} = \frac{(\gamma-1)a}{2c},\ \frac{\nu_3}{\nu_1} = \frac{(\gamma-1)b_2^2a^2}{2cB_2(a^2-b_1^2)} \\ \frac{\nu_4}{\nu_1} = -\frac{c}{2\gamma c_v},\ \frac{\nu_5}{\nu_1} = \frac{-(\gamma-1)B_1b_2^2a}{2cB_2(a^2-b_1^2)}\end{aligned}\right\} \quad (11)$$

The remaining five equations of the characteristic system are obtained by substituting (11) into (9) for the three cases of interest, viz., $a = a_f, a_s, 0$. This gives

$$a_f u_\beta + 2cc_\beta/(\gamma-1) + (a_f^2 - c^2)B_{2\beta}/B_2 - B_1(a_f^2 - c^2)v_\beta/a_fB_2 - c^2s_\beta/\gamma(\gamma-1)c_v = 0 \quad (12)$$

$$-a_f u_\alpha + 2cc_\alpha/(\gamma-1) + (a_f^2 - c^2)B_{2\alpha}/B_2 + B_1(a_f^2 - c^2)v_\alpha/a_fB_2 - c^2s_\alpha/\gamma(\gamma-1)c_v = 0 \quad (13)$$

$$a_s u_\eta + 2cc_\eta/(\gamma-1) + (a_s^2 - c^2)B_{2\eta}/B_2 - B_1(a_s^2 - c^2)v_\eta/a_fB_2 - c^2s_\eta/\gamma(\gamma-1)c_v = 0 \quad (14)$$

$$-a_s u_\zeta + 2cc_\zeta/(\gamma-1) + (a_s^2 - c^2)B_{2\zeta}/B_2 + B_1(a_s^2 - c^2)v_\zeta/a_sB_2 - c^2s_\zeta/\gamma(\gamma-1)c_v = 0 \quad (15)$$

$$s_\zeta = 0 \quad (16)$$

The characteristic form of the system (1)–(5) consists of (8) and (12)–(16). For the case of a transverse field only, i.e., $B_1 = 0$, $a_s = 0$ and $a_f = \omega = [b_2^2 + c^2]^{1/2}$, the characteristic system reduces exactly to that previously presented.[4–6]

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The Free-Molecule Impact-Pressure Probe of Arbitrary Length†

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This note is written to supplement the work recently published by Pond,[1] who calculated the probabilities of molecules passing through cylindrical tubes for selected values of entrance velocity and tube length. The results reported here agree with those of Pond's where comparison is possible, but were obtained by a completely independent and different analysis of the underlying theory late in 1961. Our analysis of the free-molecule impact-pressure probe was based on a direct numerical integration of the relevant equations set up by Harris and Patterson.[2] For completeness these are stated below. The calculations were performed with the aid of an IBM-650 computer for specific speed ratios between 0 and 20 and for tube geometries ranging continuously from an orifice to an infinitely long tube.

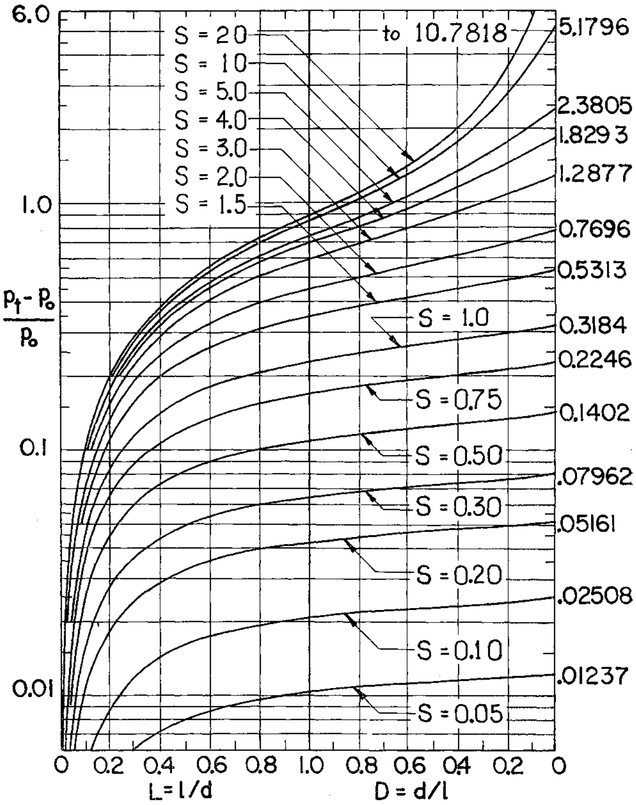

Fig. 1. Fractional deviation of impact-tube pressure from orifice gauge pressure, $(p_t - p_0)/p_0$, as a function of tube geometry.

The gage volume pressure p_t behind an impact tube may be related to the free-stream static pressure p_1 by means of probability functions W, which depend on the speed ratio S of the flow and the diameter-to-length ratio D of the tube:

$$p_t\sqrt{T_1}/p_1\sqrt{T_g} = W(S,D)/W(0,D) \quad (1)$$

In this expression T_1 and T_g are the free-stream and gage temperatures, respectively, and

$$\left.\begin{aligned}
W(S,D) &= \alpha\left[2\chi(S) - e^{-S^2}\psi(D) - \frac{4S}{\sqrt{\pi}}\eta(S,D)\right] + (1-2\alpha)\left[\chi(S) - e^{-S^2}\zeta(D) - \frac{4S}{\sqrt{\pi}}\mu(S,D)\right] \\
\chi(S) &= e^{-S^2} + S\sqrt{\pi}(1 + \operatorname{erf} S) \\
\psi(D) &= 2(\sqrt{1+D^2} - 1)/D^2 \\
\zeta(D) &= 2[(1+D^2)^{3/2} - D^3 - 1]/3D^2 \\
\eta(S,D) &= \frac{1}{D}\int_0^1 dY \times \int_0^{\sqrt{(1-Y^2)/(1-Y^2+1/D^2)}} [1 + \operatorname{erf}(S\sqrt{1-t^2})]e^{-S^2t^2}dt \\
\mu(S,D) &= \int_0^1 \eta(S,D/X)dX
\end{aligned}\right\} \quad (2)$$

The quantity $\alpha(D)$ is the Clausing[4] probability function appearing in

$$w(X,D) = \alpha + (1-2\alpha)X \quad (3)$$

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which expresses the probability w that a molecule, leaving the wall at position $X = x/l$ from the inlet, will reach the outlet of the tube.

The limiting case of the infinitely long tube was analyzed separately, because a direct numerical integration of Eq. (2) proved increasingly inaccurate as $D \to 0$. It can be shown that[3]

$$\frac{p_{t\infty}}{p_1}\sqrt{\frac{T_1}{T_g}} = e^{-S^2} + \frac{\sqrt{\pi}S}{2}\left(1 + \text{erf } S - \frac{6}{\pi}\lim_{D\to 0}\frac{d\mu}{dD}\right) \quad (4)$$

where

$$\lim_{D\to 0}\frac{d\mu}{dD} = \int_0^\infty \frac{\eta(S,D/X) - \eta(S,O)}{(D/X)^2}\, d(D/X) \quad (5)$$

To make Eq. (5) suitable for numerical integration, the approximation

$$\lim_{D\to 0}\frac{d\mu}{dD} = -\int_0^{(D/X)_I} \frac{\frac{\pi}{4}(1 + \text{erf } S) - \eta(S,D/X)}{(D/X)^2}\, d(D/X) - \frac{\pi(1 + \text{erf } S)}{4(D/X)_I} \quad (6)$$

proves useful, since the integrand in this equation becomes a known function[3] of S at $D/X = 0$. The accuracy attainable with Eq. (6) increases with increasing values of $(D/X)_I$. Fig. 1 shows the calculated results for the infinitely long tube using $(D/X)_I = 220$.

For convenience of representing the data, the quantity $p_t\sqrt{T_1}/p_1\sqrt{T_g}$ has been divided by the orifice function $\chi(S) = p_0\sqrt{T_1}/p_1\sqrt{T_g}$. The portion of the resulting ratio which is above unity, $(p_t - p_0)/p_0$, has been plotted in Fig. 1.

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Pure Bending, Twisting, and Stretching of Skewed, Heterogeneous, Aeolotropic Plates

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The following discussion is concerned with the problems of pure bending, twisting, and stretching in the theory of heterogeneous aeolotropic plates. Previous results[1] for rectangular laminated plates will be extended to skewed heterogeneous plates.

Formulation of the Problem

Consider a thin elastic plate of constant thickness that is heterogeneous in the thickness direction z. Let x,y be the coordinates in the undeflected bottom face of the plate ($z = 0$). The general theory of bending and stretching of such a plate was established in Refs. 1 and 2 and is used in the present development.

The expressions for stress couples in terms of stress resultants and plate curvatures κ are of the form

$$[M] = [C^*][N] + [D^*][\kappa] \quad (1)$$

where the C^*, D^* matrices are defined by Eq. (14) of Ref. 2.

We consider heterogeneous plates bounded by two straight lines $x = \pm 1$ and by two straight lines $y = \pm {}^1/_2 c - xtg\alpha$. Angle α is the angle of skew of the plate, 21 is the span of the plate, and c is the chord of the plate.

Expressions for bending moment M_n and twisting moment M_{nt} acting along the edges $y = \pm {}^1/_2 c - xtg\alpha$ follow by means of the usual transformation equations for plate stress couples, which hold for homogeneous, as well as heterogeneous, plates:

$$M_n = M_y \cos^2\alpha + M_x \sin^2\alpha + 2M_{xy}\cos\alpha\sin\alpha \quad (2)$$

$$M_{nt} = (M_x - M_y)\cos\alpha\sin\alpha + M_{xy}(\cos^2\alpha - \sin^2\alpha) \quad (3)$$

Within the framework of this theory,[1, 2] which does not account for transverse effects, there occur concentrated forces P at the corners of the plate given by

$$P = (M_{xy} + M_{nt})_{corner} \quad (4)$$

It was shown in Ref. 1 that for the problems of uniform distributions of stress resultants and couples, with which we are concerned, it is sufficiently general to assume a deflection function

$$w = {}^1/_2 k_x x + {}^1/_2 k_y y + k_{xy} xy \quad (5)$$

where k_x, k_y, k_{xy} are the constant values of the plate curvatures κ_x, κ_y, κ_{xy}, respectively.

Pure Bending of Heterogeneous Skewed Plate

In this case the following boundary conditions must be satisfied:

$$\text{at } x = \pm 1: cM_x = M_1,\ N_x = 0,\ N_{xy} = 0 \quad (6)$$

$$\text{at } y = \pm {}^1/_2 c - xtg\alpha: \quad (2l/\cos\alpha)M_n = M_2,\ N_n = 0,\ N_{nt} = 0 \quad (7)$$

$$\text{at the corners:} \quad M_{xy} + M_{nt} = 0 \quad (8)$$

Using Eqs. (1)–(3) and (6)–(8), we find for the constants in Eq. (5) the following system of three algebraic equations:

$$\begin{bmatrix} \mathfrak{D}_{11} & \mathfrak{D}_{12} & \mathfrak{D}_{13} \\ \mathfrak{D}_{21} & \mathfrak{D}_{22} & \mathfrak{D}_{23} \\ \mathfrak{D}_{31} & \mathfrak{D}_{32} & \mathfrak{D}_{33} \end{bmatrix}\begin{bmatrix} k_x \\ k_y \\ k_{xy} \end{bmatrix} = \begin{bmatrix} M_x \\ M_n \\ 0 \end{bmatrix} \quad (9)$$

where

$$\mathfrak{D}_{11} = D_{xx}^*,\ \mathfrak{D}_{12} = D_{xy}^*,\ \mathfrak{D}_{13} = D_{xs}^* \quad (10)$$

$$(\mathfrak{D}_{21}, \mathfrak{D}_{22}, \mathfrak{D}_{23}) = (D_{xy}^*, D_{yy}^*, D_{ys}^*)\cos^2\alpha + (D_{xx}^*, D_{xy}^*, D_{xs}^*)\sin^2\alpha + 2(D_{xs}^*, D_{ys}^*, D_{ss}^*)\cos\alpha\sin\alpha \quad (11)$$

$$(\mathfrak{D}_{31}, \mathfrak{D}_{32}, \mathfrak{D}_{33}) = ([D_{xx}^* - D_{xy}^*], [D_{xy}^* - D_{yy}^*], [D_{xs}^* - D_{ys}^*]) \cos\alpha\sin\alpha + 2(D_{xs}^*, D_{ys}^*, D_{ss}^*)\cos^2\alpha \quad (12)$$

The solution of Eqs. (9) is compactly written as

$$[k] = [\mathfrak{D}^{-1}][M] \quad (13)$$

Using Eqs. (5) and (13) together with Eq. (1), and noting that all stress resultants vanish throughout the plate, we obtain the couples expressed in the x-y coordinates. Eqs. (61) and (70) of Ref. 1 give, then, the components of strain and stress, respectively.

Twisting of Heterogeneous Skewed Plates

The following boundary conditions prevail:

$$\text{at } x = \pm 1: M_x = 0,\ N_x = 0,\ N_{xy} = 0 \quad (14)$$

$$\text{at } y = \pm {}^1/_2 c - xtg\alpha: M_n = 0,\ N_n = 0,\ N_{nt} = 0 \quad (15)$$

In addition to this, we have the applied torque given by

$$T = cP \quad (16)$$

Following the same procedure as above, we find for the constants in Eq. (5)

$$[k] = [\mathfrak{D}^{-1}]\begin{bmatrix} 0 \\ 0 \\ T/c \end{bmatrix} \quad (17)$$

The method for determining stresses and strains is exactly as indicated in the case of pure bending.